L^r-Helmholtz-Weyl decomposition in 3D exterior domains and its application to the Navier-Stokes equations

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First, we show that in 3D exterior domains Ω with the compact smooth boundary $\partial\Omega$, two spaces $X^r_{\mathrm{har}}(\Omega)$ and $V^r_{\mathrm{har}}(\Omega)$ of L^r-harmonic vector fields $\boldsymbol{h}$ with $\boldsymbol{h}\cdot\boldsymbol{\nu}|_{\partial\Omega}=0$ and $\boldsymbol{h}\times\boldsymbol{\nu}|_{\partial\Omega}=0$ are both of finite dimensions, where $\boldsymbol{\nu}$ denotes the unit outward normal to $\partial\Omega$. Next, we prove that for every L^r-vector field $\boldsymbol{u}$ in Ω, there exist $\boldsymbol{h}\in X^r_{\mathrm{har}}(\Omega)$, $\boldsymbol{w}\in \dot{H}^{1,r}(\Omega)^3$ with div $\boldsymbol{w}=0$ and $p\in \dot{H}^{1,r}(\Omega)$ such that $\boldsymbol{u}$ is uniquely decomposed as

$$\boldsymbol{u}=\boldsymbol{h}+\mathrm{rot}\ \boldsymbol{w}+\nabla p.$$

On the other hand, if for the given L^r-vector field $\boldsymbol{u}$ we choose its harmonic part $\boldsymbol{h}$ from $V^r_{\mathrm{har}}(\Omega)$, then we have a similar decomposition to above, while the unique expression of $\boldsymbol{u}$ holds only for $1<r<3$. Furthermore, the choice of p in $\dot{H}^{1,r}(\Omega)$ is determined in accordance with the threshold $r=3/2$.

In particular, we consider an exterior domain $\Omega\subset\mathbb{R}^3$ having compact boundary $\partial\Omega=\bigcup_{j=1}^{L}\Gamma_j$, where $\Gamma_1,\cdots,\Gamma_L$ are L disjoint smooth closed surfaces. As an application of our L^r-decomposition of vector fields, we prove the existence of weak solutions $\boldsymbol{v}$ of the stationary Navier-Stokes equations in Ω satisfying $\boldsymbol{v}|_{\Gamma_j}=\boldsymbol{\beta_j}$, $j=1,\cdots,L$ and $\boldsymbol{v}\to\boldsymbol{0}$ as $|x|\to\infty$, where $\boldsymbol{\beta}_j$, $j=1,\cdots,L$ are the given boundary data.

Our results are based on the joint work with Matthias Hieber, Anton Seyferd(TU Darmstadt), Senjo Shimizu(Kyoto Univ.) and Taku Yanagisawa(Nara Women Univ.).

References

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